Problem 1.45

Evaluate the following integrals:

(a) $\int_{-2}^{2}(2x+3)\,\delta(3x)\,dx$.

(b) $\int_{0}^{2}(x^3+3x+2)\,\delta(1-x)\,dx$.

(c) $\int_{-1}^{1}9x^2\delta(3x+1)\,dx$.

(d) $\int_{-\infty}^{a}\delta(x-b)\,dx$.

Solution

The delta function has several properties.

$$(1)\quad \int_a^b f(x)\delta(x-c)\,dx = \begin{cases} 0 & \text{if } c \notin (a,b) \\ f(c) & \text{if } c \in (a,b) \end{cases}$$

$$(2)\quad \delta(kx) = \frac{1}{|k|}\delta(x)$$

$$(3)\quad \delta(x) = \frac{d\theta}{dx}, \quad \text{where } \theta(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Use them to evaluate the integrals.

$$\begin{aligned} \int_{-2}^{2}(2x+3)\delta(3x)\,dx &= \int_{-2}^{2}(2x+3)\left[\frac{1}{|3|}\delta(x)\right]dx \\ &= \int_{-2}^{2}\frac{2x+3}{3}\delta(x-0)\,dx \\ &= \left.\frac{2x+3}{3}\right|_{x=0} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \int_{0}^{2}(x^3+3x+2)\delta(1-x)\,dx &= \int_{0}^{2}(x^3+3x+2)\left[\frac{1}{|-1|}\delta(x-1)\right]dx \\ &= \int_{0}^{2}(x^3+3x+2)\delta(x-1)\,dx \\ &= \left.(x^3+3x+2)\right|_{x=1} \\ &= 1+3+2 \\ &= 6 \end{aligned}$$

Evaluate the remaining integrals.

$$\begin{aligned}
\int_{-1}^{1} 9x^2\delta(3x+1)\,dx &= \int_{-1}^{1} 9x^2\left[\frac{1}{|3|}\delta\left(x+\frac{1}{3}\right)\right]dx \\
&= \int_{-1}^{1} 3x^2\delta\left(x+\frac{1}{3}\right)dx \\
&= (3x^2)\Big|_{x=-1/3} \\
&= 3\left(-\frac{1}{3}\right)^2 \\
&= \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{a} \delta(x-b)\,dx &= \int_{-\infty}^{a-b} \delta(u)\,du \\
&= \int_{-\infty}^{a-b} \frac{d\theta}{du}\,du \\
&= \theta(a-b) - \underbrace{\theta(-\infty)}_{=0} \\
&= \begin{cases} 0 & \text{if } a-b \leq 0 \\ 1 & \text{if } a-b > 0 \end{cases}
\end{aligned}$$